W_∞ Algebra and Geometric Formulation of QCD_2

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Gauge theories and string theory have a long standing symboitic relationship [1]. In this talk we summarize an application of some developments in 2-dim. string theory to 2-dim. QCD [2]. These developments are related to W_∞ algebra and its presence in QCD_2 unravels the structure of its non-linear gauge invariant phase space. In this framework we will derive 't Hooft's equation [3] in a geometric setting.
The model is a $U(N)$ gauge theory in 2 space-time dimensions, with fermions in the fundamental representation of $U(N)$. Choose the light cone gauge $A_+(x^-, x^+) = 0$ and consider the gauge invariant operators, at fixed $x^+(\text{light} - \text{cone time})$, involving fermions of a definite chirality

$$M(x^-, y^-; x^+) = \frac{1}{N} \sum_{a,b=1}^{N} \psi_-^a(x^-, x^+) \left(e^{i \int_x^{x^-} A_-(z, x^+) dz} \right)_{ab} \psi^{+b}(y^-, x^+) \ \ldots \tag{1}$$

In a Hamiltonian formulation at time $x^+ = 0$, we can choose the gauge $A_-(x^-, 0) = 0$. Then (1) becomes the bilocal operator $M(x, y) = \frac{1}{N} \sum_{a=1}^{N} \psi_-^a(x)\, \psi_-^{+a}(y)$. Henceforth we will understand x and y as light-cone coordinates x^- and y^-. Our first observation is that $M(x, y)$ satisfies the W_∞ algebra

$$[M(x, y),\ M(x', y')] = \delta(x - y')\, M(x', y) - \delta(x' - y)\, M(x, y'). \tag{2}$$

These are the 'Poisson brackets' in the gauge invariant *phase space*. The second point is that the phase space is non-linear in the zero charge sector

$$\int_{-\infty}^{+\infty} M(x, z)\, M(z, y)\, dz = M(x, y), \int_{-\infty}^{+\infty} M(x, x)\, dx = c \tag{3}$$

The constant c is related to the baryon number. And the third point is that the Hamiltonian can be expressed entirely in terms of $M(x, y)$:

$$\begin{aligned} H &= N \int dx dy \left[\frac{g^2}{4}\, M(x, y)\, \tilde{M}(x, y) - \frac{im^2}{4}\, S(x, y)\, M(x, y) \right] \\ \tilde{M}(x, y) &= |x - y|\, M(x, y) \ \text{ and } \ S(x, y) = \text{sgn}(x - y) \end{aligned} \tag{4}$$

The action and the path integral can be constructed using the method of co-adjoint orbits [4] or the method of W_∞ coherent states [5]

$$S = N \Big[2i \int_\Sigma ds dx^+ Tr\left(M \left[\partial_+ M, \partial_s M \right] \right) - \int_{-\infty}^{+\infty} dx^+ Tr \left(\frac{1}{4}\, im^2 S M + \frac{1}{4}\, g^2 M \tilde{M} \right) \Big] \tag{5}$$

In (5) M has been treated as an operator with matrix elements $M(x, y)$ and Σ is the half-plane: $(s, x^+) \in (-\infty, 0) \otimes (-\infty, +\infty)$, with boundary conditions, $M(x, y, x^+, s = 0) = M(x, y, x^+)$ and $M(x, y, x^+, s = -\infty) = 0$. The phase flow is described the equation of motion

$$i \partial_+ M = (im^2/8)[M, S] + (g^2/4) \left[M, \widetilde{M} \right] \tag{6}$$

A systematic perturbation theory in $\frac{1}{N}$ can be constructed using (3),(5) and (6). In the zero baryon sector, the classical solution of (3), (6), in the momentum basis is

$$M_0(k, k') = \theta(k) \delta(k - k') \tag{7}$$

This solution is related to the filling of the fermi sea upto the Lorentz invariant (light cone) fermi level $k_F = 0$. It specifies a co-adjoint representation of W_∞. One can say it is the 'master field' of QCD_2.

Perturbation theory around M_0 is constructed, in analogy with pion perturbation theory, by the parameterization:

$$M = e^{iW/\sqrt{N}} M_0 \, e^{-iW/\sqrt{N}} \tag{8}$$

where $e^{iW/\sqrt{N}}$ is a W_∞ group element, and $W = \sum_{k,k'} W_{kk'} t_{kk'}$. The matrices $t_{kk'}$ are the generators of W_∞ (analogous of the Pauli matrices for $SU(2)$). This parameterization preserves (3). Now the W_∞ algebra has 2 wedge sub-algebras. $W_{+\infty} = \{t_{kk'}, k > 0, k' > 0\}$ and $W_{-\infty} = \{t_{kk'}, k < 0, k' < 0\}$. These leave $M_0(k, k')$ invariant and hence the phase space is infact the coset $W_\infty / W_{+\infty} \otimes W_{-\infty}$, with co-ordinates $W^{+-}(k, k') = W(k, k')$, $k > 0$, $k' < 0$ and $W^{-+}(k, k') = W(k, k')$ $k < 0$, $k' > 0$. These co-ordinates provide a non-linear realization of the W_∞ algebra. Explicit formulas are obtained by substituting (8) in (2). The general algebra of W^{+-} and W^{-+} is highly non-linear, however to leading order in $\frac{1}{N}$ it reduces to the Heisenberg algebra

$$[W^{+-}(k, k'), W^{-+}(l, l')] = (1/2)\delta(k - l')\delta(k' - l) + o(N^{-1/2}) \; [W^{+-}, W^{+-}] = [W^{-+}, W^{-+}] = 0 + o(N^{-1/2}) \tag{9}$$

Substituting (8) into (5) and expanding generates the perturbation theory $S = NS_0 + S_1 + \frac{1}{N} S_2 \ldots$. The term S_1 gives the propagator of the fluctuations W^{-+}, W^{+-} and S_2, S_3 etc. gives the interactions.
We explicitly present the equation of motion that follows from S_1:

$$\begin{aligned} &i\partial_+ W^{-+}(k, k'; x^+) = (m^2/4)(1/k + 1/k') W^{-+}(k, k'; x^+) \\ &\quad - (g^2/4\pi) \int_k^{-k'} dp (1/p^2) \big[W^{-+}(k - p, k' + p; x^+) - W^{-+}(k, k'; x^+)\big] + o(N^{-1/2}) \end{aligned} \tag{10}$$

In (10), $k > 0$, $k' > 0$ and we also have the boundary conditions $W^{-+}(k, k' = 0) = 0$ (fermi satisfies) and $W^{-+}(k, k' = \infty) = 0$ (finite energy).
Now introducing the variables $x = \frac{k}{r_-}$, $r_- = k + k'$, $y = \frac{p_+ k'}{r_-}$ and the fourier transform $W^{-+}(k, k'; x^+) = \int \frac{dr^+}{2\pi} \phi(x; r_-, r_+) e^{ir_+ x^+}$ (10) implies the 't Hooft equation [3]

$$4r_- r_+ \phi(x) = m^2 (\frac{1}{x} + \frac{1}{1 - x}) \phi(x) - \frac{g^2}{\pi} \int_0^1 \frac{dy}{(y - x)^2} \big(\phi(y) - \phi(x)\big) \tag{11}$$

with boundary conditions $\phi(0) = \phi(1) = 0$. This leads to the well known (stringy) meson spectrum: $\phi_n(x) \sim \sin n\phi x$, $r_+ r_- \sim n$ for large n.

Baryons:

Witten had pointed out that baryons are solitons of the large N theory [6]. In the present framework we have to solve the eqn. of motion (8) with $M^2 = M$ and $Tr(1 - M) = B$. Their amplitude is proportional to e^{-N}, which is typical of stringy non-perturbative behaviour [7]. We have to hold the regularized baryon number to be non-zero. These non-trivial classical solutions will correspond to different self-consistent Hartree-Fock potentials which arise out of populating quasi-particle wave function above the Dirac sea. The important point here is that these classical solutions are given by a function $M_{c\ell}(k^-, k^{-'}, t)$ of 2 variables: $k^- \pm k^{-'}$. If we call the conjugate variables Y and X, then the fourier transform of $M_{c\ell}$ is $M_{c\ell}(Y, X)$. X represents a centre of mass type co-ordinate. The variable Y does not seem to have an analogue in point particle theories. It indicates the possibility that the baryon is itself a *stringy* state which appears particle-like only at long wave lengths. One then expects to introduce 2 collective co-ordinates parameterized by $(\tau, \sigma) : M = m_{c\ell}(Y - y(\sigma, \tau), X - x(\sigma, \tau))$. One wonders whether these remarks on stringy solitons have any bearing to those in [8].

Acknowledgement

I acknowledge A. Dhar and G. Mandal for many discussions on this subject.

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